A Survey For Some Special Curves In Isotropic Space $\mathbf{I}_3^1$

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Abstract

In this paper, curves of AW(k)-type in isotropic space $\mathbf{I}_3^1$ are defined. Using Frenet frames in isotropic space $\mathbf{I}_3^1$, curvature conditions of AW(k)-type curves are given. In addition, new characterizations of Bertrand and Mannheim curves are obtained.

Keywords: Isotropic space, Frenet frame, Bertrand curve, Mannheim curve, Curvature, Torsion.

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1. Introduction.

The assumption that our universe is homogeneous and isotropic means that its evolution can be represented as a time-ordered sequence of three-dimensional space-like hypersurfaces, each of which is homogeneous and isotropic. These hypersurfaces are the natural choice for surfaces of constants time.

Homogeneity means that the physical conditions are the same at every point of any given hypersurface. Isotropy means that the physical conditions are identical in all directions when viewed from a given point on the hypersurface. Isotropy at every point automatically enforces homogeneity. However, homogeneity does not necessarily imply isotropy.

Homogeneous and isotropic spaces have the largest possible symmetry group; in three dimensions there are three independent translations and three rotations. These symmetries strongly restrict the admissible geometry for such spaces. There exist only three types of homogeneous and isotropic spaces with simple topology: (a) flat space, (b) a three-dimensional sphere of constant positive curvature, and (c) a three-dimensional hyperbolic space of constant negative curvature [7].

Many interesting results on curves of AW(k)-type have been obtained by many mathematicians (see [1], [3], [4], [5], [6]). Also, Bertrand curves have been studied in [8] and [11].

In this paper, we have done a study about some special curves in Isotropic Space I_3^1. However, to the best of author's knowledge, Bertrand and Mannheim curves of AW(k)-type has not been presented in Isotropic Space I_3^1. Thus, the study is proposed to serve such a need.

Our paper is organized as follows. In section 2, the basic notions and properties of a Frenet curve are reviewed. In section 3, we study curves of AW(k)-type in Isotropic Space I_3^1. We also study Bertrand and Mannheim curves of AW(k)-type in section 4.

2. Basic notions and properties

Let $\alpha : I \to I_3^1$, $I \subset IR$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t)),$$

where $x(t)$, $y(t)$, $z(t) \in C^3$ (the set of three times continuously differentiable functions) and t run through a real interval [9].

Let α be a curve in I_3^1, parameterized by arc length $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s)). \tag{1}$$

Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by [9]

$$\begin{aligned} \kappa(s) &= x^{'} y^{''} - y^{'} x^{''} \\ \tau(s) &= \frac{\det(\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s))}{\kappa^2(s)} \end{aligned} \tag{2}$$

and associated moving trihedron is given by

$$\begin{aligned} t(s) &= \alpha^{'}(s) \\ n(s) &= \frac{1}{\kappa(s)} \alpha^{''}(s) \\ b(s) &= (0, 0, 1) \end{aligned} \tag{3}$$

The vectors t_α, n_α, b_α are called the vectors of the tangent, principal normal and binormal line of α, respectively. For their derivatives the following Frenet formulas hold

$$
\begin{aligned}
t^{'}(s) &= \kappa(s)n(s) \\
n^{'}(s) &= -\kappa(s)t(s)+\tau(s)b(s) \\
b^{'}(s) &= 0
\end{aligned} \tag{4}
$$

Scalar product in the Isotropic space I_3^1 is defined by

$$
< X, Y >= x_1y_1 + x_2y_2 \tag{5}
$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. If $x_1y_1 + x_1y_1 = 0$, then $< X, Y >= x_3y_3$.

The isotropic norm of a vector $X = (x_1, x_2, x_3)$ is defined by

$$
\|X\| = \left\|\widetilde{X}\right\| = \sqrt{x_1^2 + x_2^2}
$$

where $\sim$ on the vector denotes the canonical projection of the vector to the base plane $x_3 = 0$. If $\|X\| = 0$, i.e. if X is an isotropic vector, then the sumplementary invariant called range of the vector X is introduced

$$
[X] = x_3.
$$

If $\|X\| \neq 0$, then X called Euclidean vector. [10]

From now on in calculations, " $\widetilde{X}$ " canonical projection of the vectors are denoted as " X ".

Proposition 2.1. Let α be a Frenet curve of I_3^1 of osculating order 3 then we have

$$
\begin{aligned}
\alpha^{'}(s) &= t(s) \\
\alpha^{''}(s) &= t^{'}(s) = \kappa(s)n(s) && (6) \\
\alpha^{'''}(s) &= -\kappa^2(s)t(s) + \kappa^{'}(s)n(s) + \kappa(s)\tau(s)b(s) && (7) \\
\alpha^{IV}(s) &= -3\kappa(s)\kappa^{'}(s)t(s) + [\kappa^{''}(s) - \kappa^3(s)]n(s) \\
&\quad +[2\kappa^{'}(s)\tau(s) + \kappa(s)\tau^{'}(s)]b(s) && (8)
\end{aligned}
$$

Notation. Let us write

$$\begin{aligned} N_1(s) &= \kappa(s)n(s) && (9)\\ N_2(s) &= \kappa^{'}(s)n(s) + \kappa(s)\tau(s)b(s) && (10)\\ N_3(s) &= [\kappa^{''}(s) - \kappa^3(s)]n(s) + [2\kappa^{'}(s)\tau(s) + \kappa(s)\tau^{'}(s)]b(s) && (11) \end{aligned}$$

Corollary 2.2. $\alpha^{'}(s)$, $\alpha^{''}(s), \alpha^{'''}(s)$ and $\alpha^{IV}(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$ and $N_3(s)$ are linearly dependent.

Theorem 2.3. Let α be a Frenet curve of I_3^1 of osculating order 3 then

$$N_3(s) =< N_3(s), N_1^*(s) > N_1^*(s)+ < N_3(s), N_2^*(s) > N_2^*(s) \qquad (12)$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, N_2^*(s) = \frac{N_2(s)- < N_2(s), N_1^*(s) > N_1^*(s)}{\|N_2(s)- < N_2(s), N_1^*(s) > N_1^*(s)\|}. \qquad (13)$$

3. Curves of AW(k)-type.

Definition 3.1. Frenet curves (of osculating order 3) are [1]

i) of type weak AW(2) if they satisfy

$$N_3(s) =< N_3(s), N_2^*(s) > N_2^*(s), \qquad (14)$$

ii) of type weak AW(3) if they satisfy

$$N_3(s) =< N_3(s), N_1^*(s) > N_1^*(s). \qquad (15)$$

Proposition 3.2. Let α be a Frenet curve of order 3. If α is of type weak AW(2) then

$$\kappa^{''}(s) - \kappa^3(s) = 0. \qquad (16)$$

Corollary 3.3. Let α be a Frenet curve of type weak AW(2). If α is a plane curve then

$$\kappa(s) = \mp\frac{\sqrt{2}}{s+c}; c = const. \qquad (17)$$

Proposition 3.4. Let α be a Frenet curve of order 3. If α is of type weak AW(3) then

$$2\kappa^{'}(s)\tau(s) + \kappa(s)\tau^{'}(s) = 0. \qquad (18)$$

Corollary 3.5. Let α be a Frenet curve of type weak AW(3). Then

$$\tau(s) = \frac{c}{\kappa^2(s)}; c = const. \tag{19}$$

Definition 3.6. Frenet curves are (see [1])
i) of type AW(1) if they satisfy

$$N_3(s) = 0, \tag{20}$$

ii) of type AW(2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) =< N_3(s), N_2(s) > N_2(s), \tag{21}$$

iii) of type AW(3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) =< N_3(s), N_1(s) > N_1(s). \tag{22}$$

Theorem 3.7. Let α be a Frenet curve of order 3. Then α is of type AW(1) if and only if

$$\kappa^{''}(s) - \kappa^3(s) = 0 \tag{23}$$

and

$$\tau(s) = \frac{c}{\kappa^2(s)}; c = const. \tag{24}$$

Proof. Let α be a curve of type AW(1). From Definition 3.6. (i) $N_3(s) = 0$. then from (11) equality, we have

$$[\kappa^{''}(s) - \kappa^3(s)]n(s) + [2\kappa^{'}(s)\tau(s) + \kappa(s)\tau^{'}(s)]b(s) = 0.$$

Furthermore, since $n(s)$ and $b(s)$ are linearly independent, we get (23) and (24).

The converse statement is trivial. Hence our theorem is proved.

Corollary 3.8. Every plane curve of type AW(1) is also of type weak AW(2).

Theorem 3.9. Let α be a Frenet curve of order 3. Then α is of type AW(2) if and only if

$$2[\kappa^{'}(s)]^2\kappa(s)\tau^2(s) + \kappa^2(s)\tau(s)\kappa^{'}(s)\tau^{'}(s) + \kappa^5(s)\tau^2(s) - \kappa^{''}(s)\kappa^2(s)\tau^2(s) = 0 \tag{25}$$

and

$$2[\kappa^{'}(s)]^{3}\tau(s)+[\kappa^{'}(s)]^{2}\kappa(s)\tau^{'}(s)+\kappa^{4}(s)\kappa^{'}(s)\tau(s)-\kappa^{'}(s)\kappa^{''}(s)\kappa(s)\tau(s)=0 \tag{26}$$

Proof. If α curve is of type AW(2), (21) holds on α. Substituting (10) and (11) into (21), we have (25) and (26).

Theorem 3.10. Let α be a Frenet curve of order 3. Then α is of type AW(3) if and only if

$$2\kappa^{2}(s)\kappa^{'}(s)\tau(s)+\kappa^{3}(s)\tau^{'}(s)=0 \tag{27}$$

Proof. Since α is of type AW(3), (22) holds on α. So substituting (9) and (11) into (22), we have (27).

4. Bertrand Curves and Mannheim Curves of AW(k)-type.

In this section, we give the characterizations of Bertrand and Mannheim Curves of AW(k)-type.

Remark 4.1. Let α be a Frenet curve of order 3 of I_3^1. For $\tau(s)\neq 0$, α is a Bertrand curve if and only if there exist a linear relation

$$A\kappa(s)+B\tau(s)=1 \tag{28}$$

where A, B are non-zero constant and $\kappa(s)$ and $\tau(s)$ are the curvature functions of α [9].

Corollary 4.2. Suppose that $\kappa(s)\neq 0$ and $\tau(s)\neq 0$. Then α is a Bertrand curve if and only if there exist a non-zero real number A such that [2]

$$A[\tau^{'}(s)\kappa(s)-\kappa^{'}(s)\tau(s)]-\tau^{'}(s)=0. \tag{29}$$

Theorem 4.3. Let $\alpha : I\rightarrow I_3^1$ be a Bertrand curve with $\kappa(s)\neq 0$ and $\tau(s)\neq 0$. Then α is of type AW(2) if and only if there is a non-zero real number A such that

$$\begin{gathered}2[\kappa^{'}(s)]^{2}\kappa(s)\tau^{2}(s)+A\kappa^{3}(s)\kappa^{'}(s)\tau(s)\tau^{'}(s)-\kappa^{2}(s)[\kappa^{'}(s)]^{2}\tau^{2}(s)\\+\kappa^{5}(s)\tau^{2}(s)-\kappa^{''}(s)\kappa^{2}(s)\tau^{2}(s)=0\end{gathered} \tag{30}$$

and

$$\begin{gathered}2[\kappa^{'}(s)]^{3}\tau(s)+A\kappa^{2}(s)[\kappa^{'}(s)]^{2}\tau^{'}(s)-\kappa(s)[\kappa^{'}(s)]^{3}\tau(s)\\+\kappa^{4}(s)\kappa^{'}(s)\tau(s)-\kappa^{'}(s)\kappa^{''}(s)\kappa(s)\tau(s)=0\end{gathered} \tag{31}$$

Proof. Since α is of type AW(2), (25) and (216) holds and since α is a Bertrand curve, (29) equality holds. If both of these equations are considered, (30) and (31) are obtained.

Theorem 4.4. Let $\alpha : I \to I_3^1$ be a Bertrand curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then α is of type AW(3) if and only if

$$2\kappa^2(s)\kappa^{'}(s)\tau(s) + A\kappa^4(s)\tau^{'}(s) - \kappa^3(s)\kappa^{'}(s)\tau(s) = 0. \tag{32}$$

Proof. Now suppose that $\alpha : I \to I_3^1$ be a Bertrand curve of type AW(3) with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then the equation (27) and (29) hold on α. Thus, we get (32).

Definition 4.5. Let α be a curve in I_3^1. If its principal normal is the binormal another curve then α is called Mannheim curve in I_3^1.

Theorem 4.6. Let α be a curve in I_3^1. Then α is Mannheim curve if and only if its curvature

$$\kappa(s) = c; \quad c = const. \tag{33}$$

Proof. Let $\alpha = \alpha(s)$ be a Mannheim curve in I_3^1. Let us denote by $\{t_\alpha(s), n_\alpha(s), b_\alpha(s)\}$ the Frenet frame field of α. The curve $\overline{\alpha}(s)$ is parametrized by arclength s as

$$\overline{\alpha}(s) = \alpha(s) + c_1(s)n(s) \tag{34}$$

for some functions $c_1(s) \neq 0$. Differentiating (34) with respect to s, we find

$$\overline{\alpha}^{'}(s) = (1 - c_1(s)\kappa(s))t(s) + c_1^{'}(s)n(s) + c_1(s)\tau(s)b(s). \tag{35}$$

Since the binormal direction of $\overline{\alpha}(s)$ coincides with the principal normal of $\alpha(s)$, we have

$$c_1^{'}(s) = 0.$$

Hence $c_1(s) = const$. The second derivative $\overline{\alpha}^{''}(s)$ with respect to s is

$$\overline{\alpha}^{''}(s) = -c_1(s)\kappa^{'}(s)t(s) + [\kappa(s) - c_1(s)\kappa^2(s)]n(s) + c_1(s)\tau^{'}(s)b(s). \tag{36}$$

Since $n(s)$ is the binormal direction of $\overline{\alpha}(s)$, we have

$$\kappa(s) - c_1(s)\kappa^2(s) = 0. \tag{37}$$

From (37), we get

$$\kappa(s) = c \tag{38}$$

where $c = \frac{1}{c_1(s)}$.

Conversely, let $\overline{\alpha}(s)$ be a curve in I_3^1 with $\kappa(s) = \frac{1}{c_1(s)}$. Then the curve

$$\overline{\alpha}(s) = \alpha(s) + c_1(s)n(s)$$

has binormal direction $n(s)$. It follows that $\alpha(s)$ is a Mannheim curve which proves the theorem.

Theorem 4.7. Let α be a Mannheim curve in I_3^1. Then α is of type AW(1) if and only if

$$\tau(s) = const. \tag{39}$$

Proof. Considering Theorem 4.6. in Theorem 3.7., we get (39). Hence the proof is completed.

Theorem 4.8. Let α be a Mannheim curve in I_3^1. Then α is of type AW(2) if and only if

$$\tau(s) = 0. \tag{40}$$

Proof. Considering Theorem 4.6. in Theorem 3.9., we get (40). Hence our theorem is proved.

Theorem 4.9. Let α be a Mannheim curve in I_3^1. Then α is of type AW(3) if and only if

$$\tau(s) = const. \tag{41}$$

Proof. Considering Theorem 4.6. in Theorem 3.10., we get (41). Hence the proof is completed.

Example 4.10. Let α be a curve in I_3^1 given by

$$\alpha(u) = \left(a\cos\frac{u}{a}, a\sin\frac{u}{a}, 0\right)$$

Then we have

$$\begin{aligned} \alpha'(u) &= \left(-\sin\frac{u}{a}, \cos\frac{u}{a}, 0\right) \\ \alpha''(u) &= \left(-\frac{1}{a}\cos\frac{u}{a}, -\frac{1}{a}\sin\frac{u}{a}0\right) \end{aligned}$$

Using (2) equality, we get $\kappa(s) = \frac{1}{a}$, $\tau(s) = 0$. $\kappa(s)$ and $\tau(s)$ hold on Theorems of 3.9, 3.10, 4.3, 4.4 and 4.8.

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